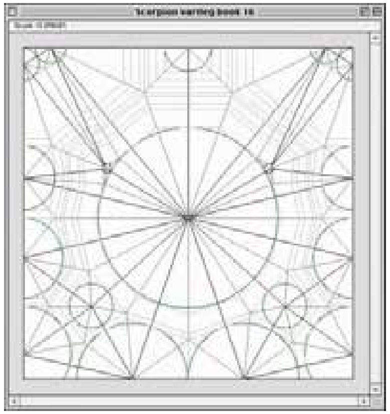
A Quick Introduction to Computational Origami



Images courtesy of Dr. Robert Lang

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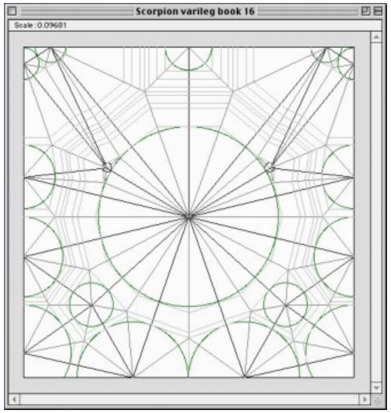
University of Illinois at Urbana-Champaign

The Power of Computation

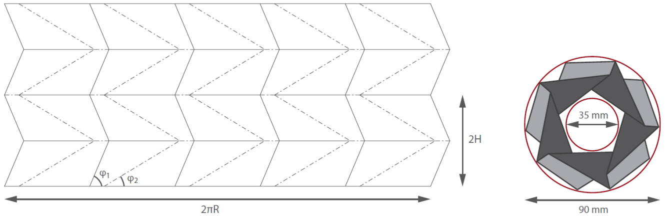
- Origami has undergone a revolution thanks to computation

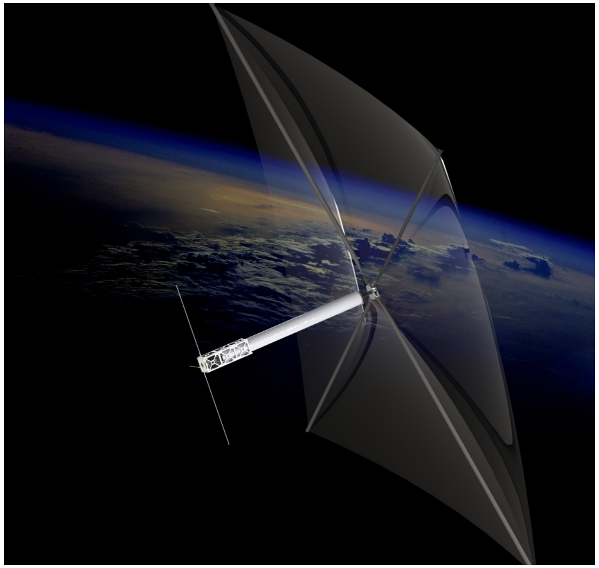


- An algorithmic approach and computational tools
  - Robert Lang: TreeMaker (can compute crease pattern for a particular class of origami base)
  - Tomohiro Tachi: Origamizer and Freeform Origami
- Enable people to create art of amazing complexity

Origami Engineering

- Origami has shockingly varied applications
  - Robotics – arms, Transformers
  - Manufacturing – sheet metal construction
  - Bio/Medical – stents, drug delivery, protein folding
  - Architecture – collapsible structures
- Satellite Sail
  - Used to de-orbit satellites
  - Uses inflatable origami mast
  - Small on delivery, large on deployment
  - Foldability, minimal material deformation, small diameter





Images courtesy of Dr. Mark Schenk

Origami Mast

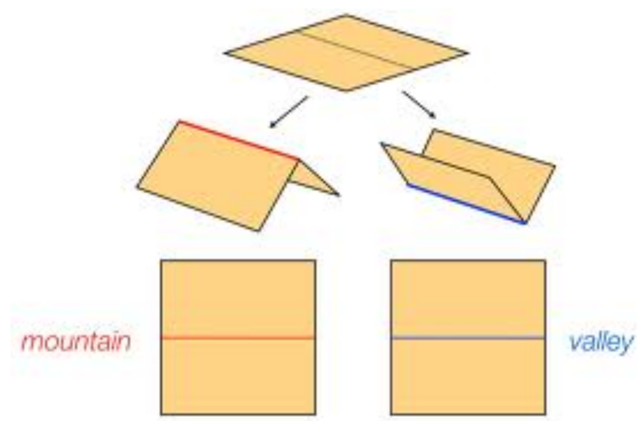
Images courtesy of Dr. Mark Schenk

What questions do we want to answer?

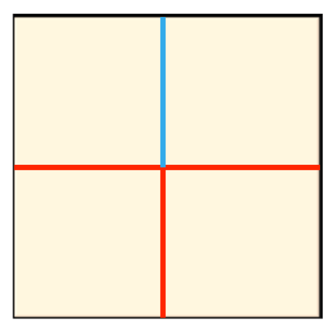
- **Foldability**: Which crease patterns can be folded?
  - Flattened into parallel layers of paper squashed into the plane
  - Can we develop an algorithm to decide this?
- **Design:** Can we find a crease pattern to generate a given final shape?
  - Efficient stick-figures [TreeMaker]
  - 2D Polygon
  - 3D Polyhedron

Mountain and Valley Folds

- Two kinds of fold
  - Mountain -
  - Valley -..

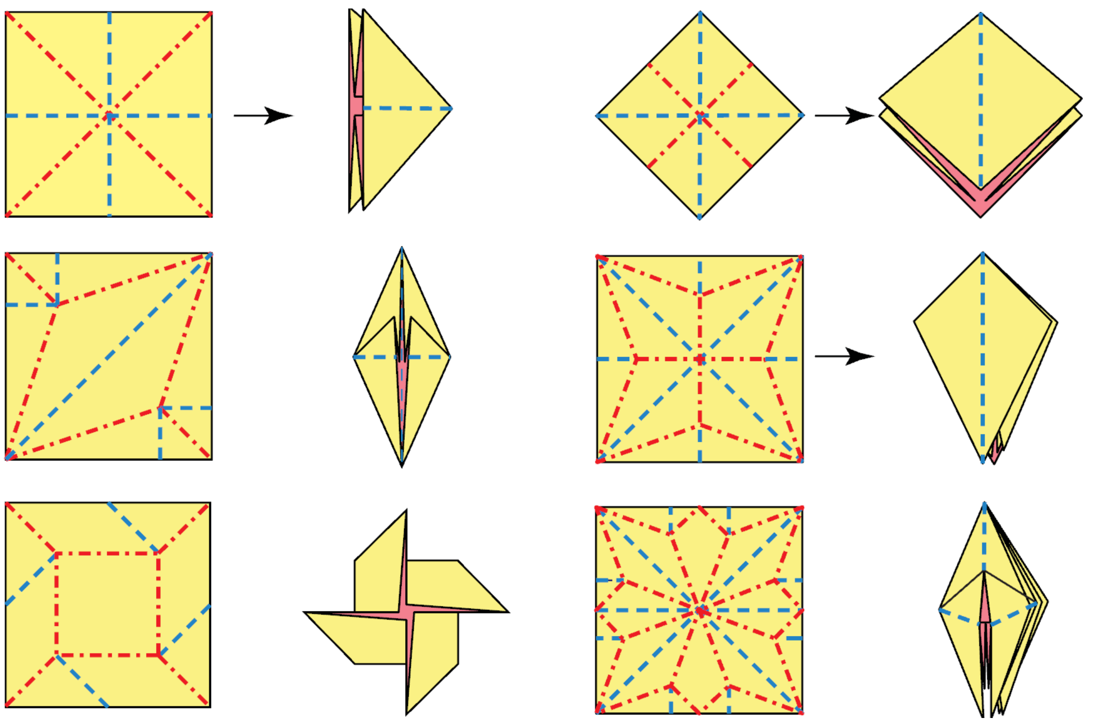


- Do some origami:



Flat Foldability

- Origami usually produces a 3D object
- In intermediate stages, it is often folded flat in the plane
- **Flat Foldability**
  Able to be configured in flat parallel sheets

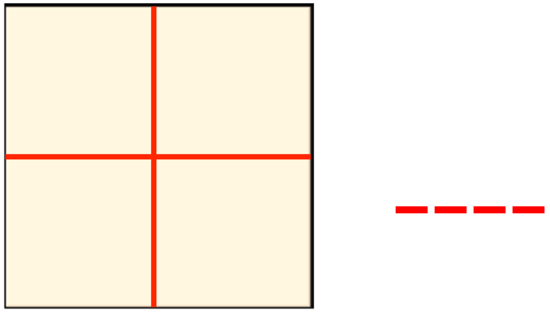


If a pattern is flat-foldable then…

- Let's look for rules that must be true if a pattern is flat-foldable
  - These are called NECESSARY conditions
  - They can't tell is if a pattern is flat-foldable (SUFFICIENT)
  - But if a pattern fails to follow the rule, then it is not flat-foldable
- Fold a flat pattern with a single vertex in the middle….what rules can you come up with?
- You have 5 minutes…..

If a pattern is flat-foldable then…

- What can we say about the number of creases going though the central point in a flat folding?
- What about this pattern?

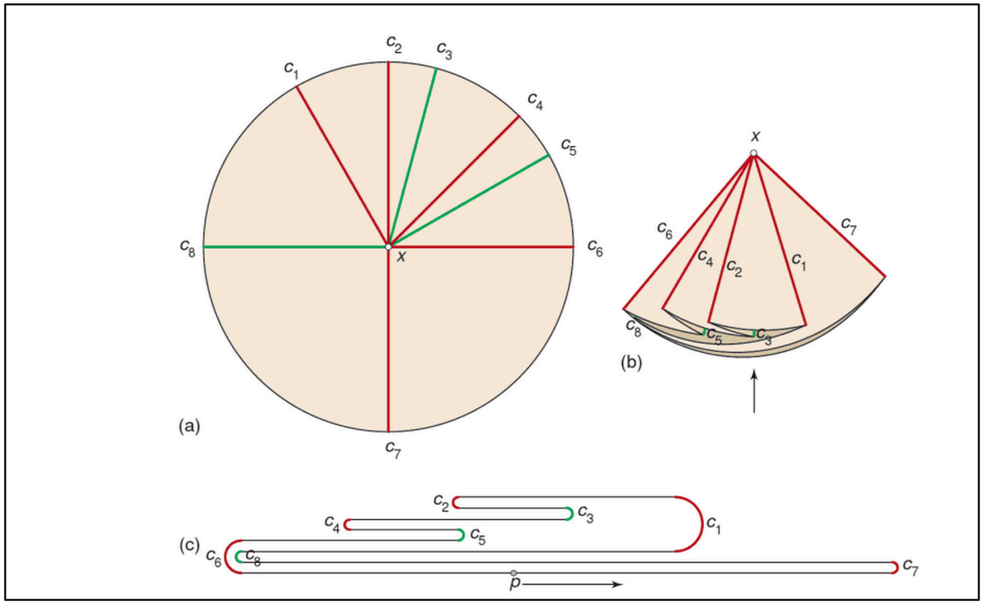


If a pattern is flat-foldable then…

- What can we say about the number of mountain creases and valley creases going though the central point in a flat folding?

The Maekawa-Justin Theorem

If M mountain creases and V valley creases meet a vertex of a flat folding then M=V+2 or V=M+2



Imagine you are bug walking around the folded paper

You start walking from point p….

The Maekawa-Justin Theorem

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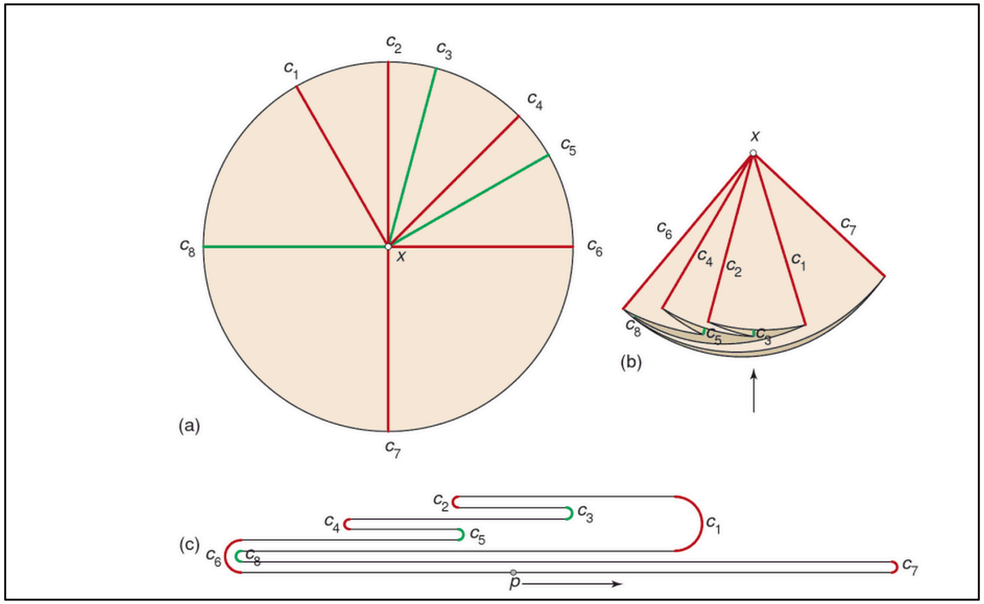
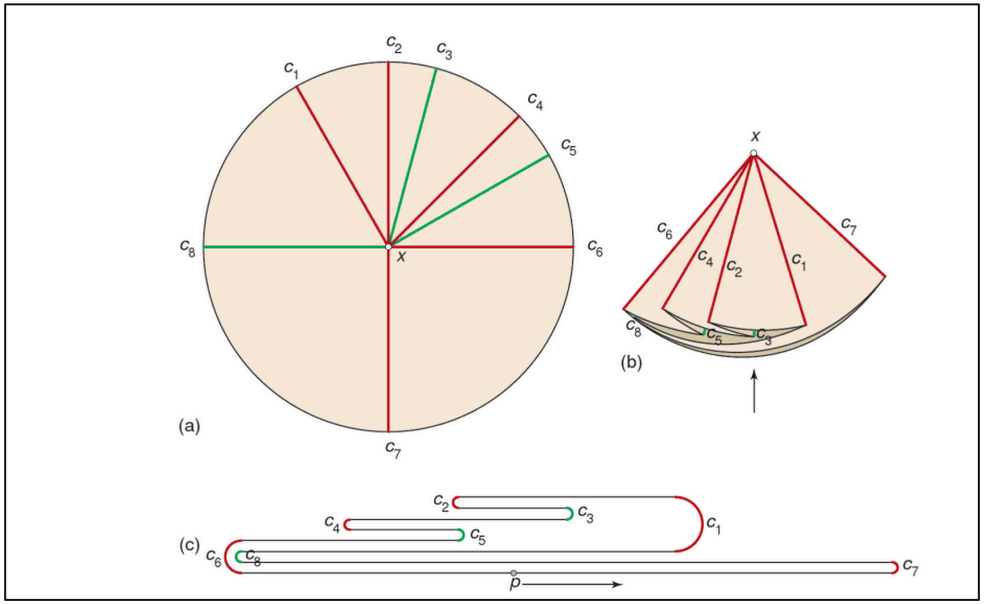


Every time you hit a Mountain fold your direction changes by how many degrees?

Every time you hit a Valley fold your direction changes by how many degrees?

The Maekawa-Justin Theorem

If M mountain creases and V valley creases meet a vertex of a flat folding then M=V+2 or V=M+2



When you get back to P, by how many degrees has your direction changed?

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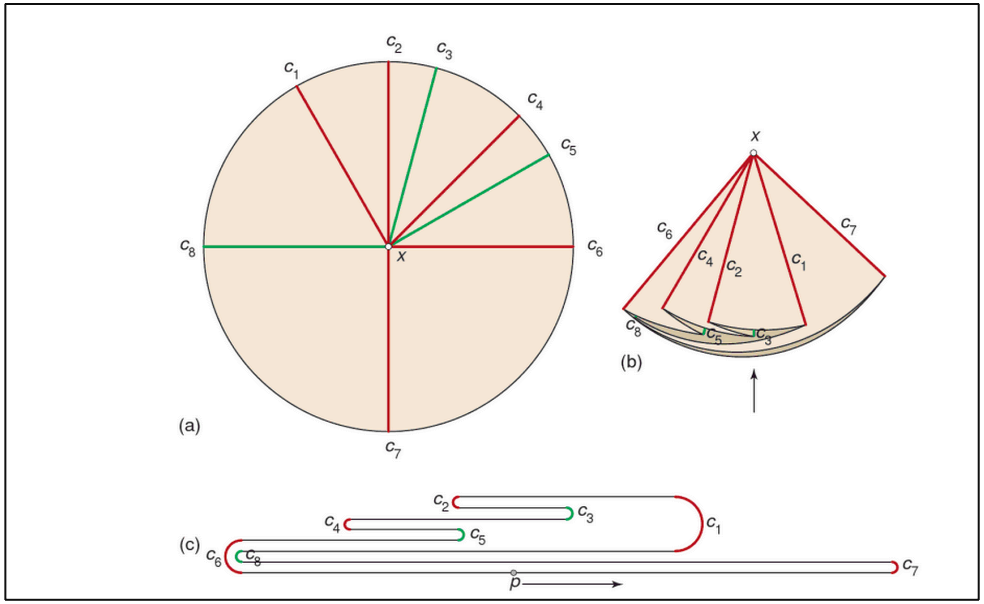

$$(180 \times M) - (180 \times V) = 360$$

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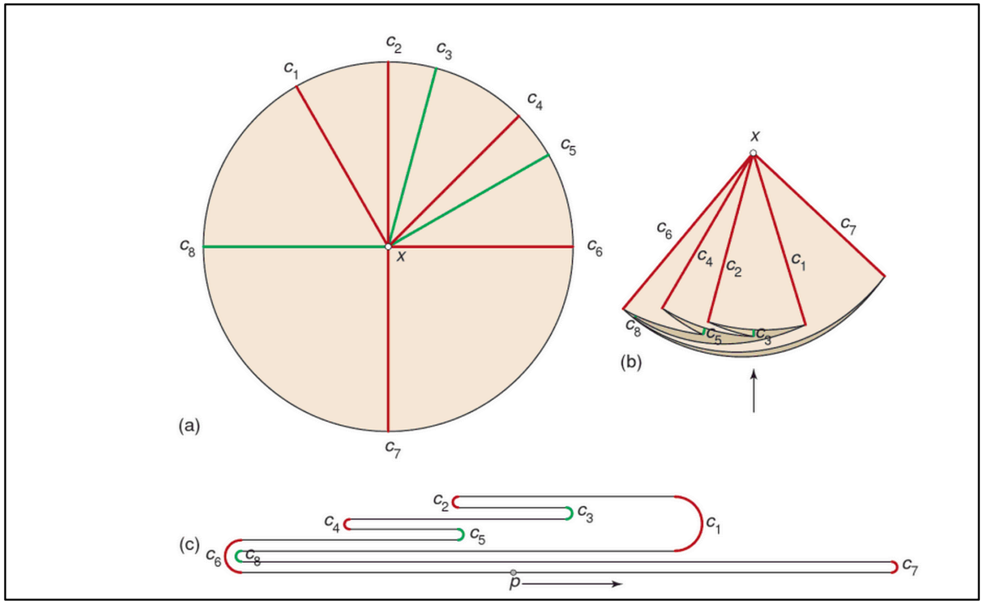

$$(180 \times M) - (180 \times V) = 360$$

$$180\,(M - V) = 360$$

The Maekawa-Justin Theorem

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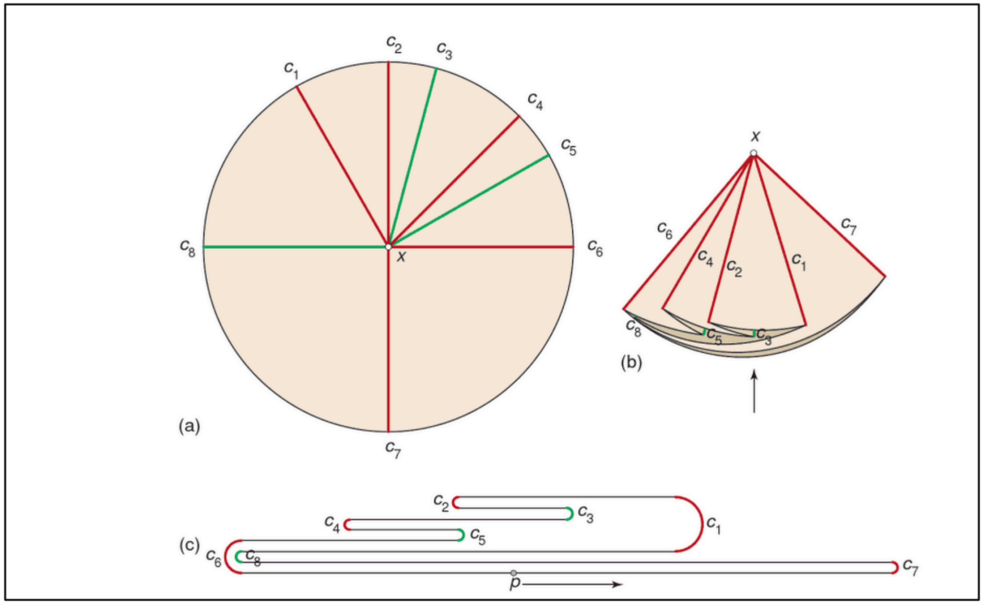


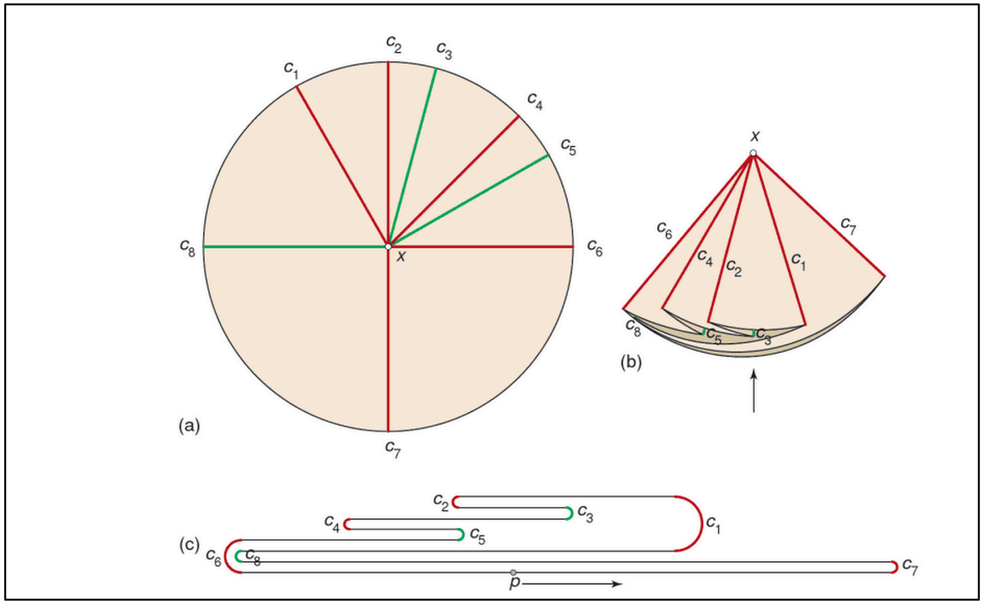
$$(180 \times M) - (180 \times V) = 360$$

$$180\,(M - V) = 360$$

$$(M-V) = 2$$

The Maekawa-Justin Theorem

If M mountain creases and V valley creases meet a vertex of a flat folding then M=V+2 or V=M+2



$$(180 \times M) - (180 \times V) = 360$$

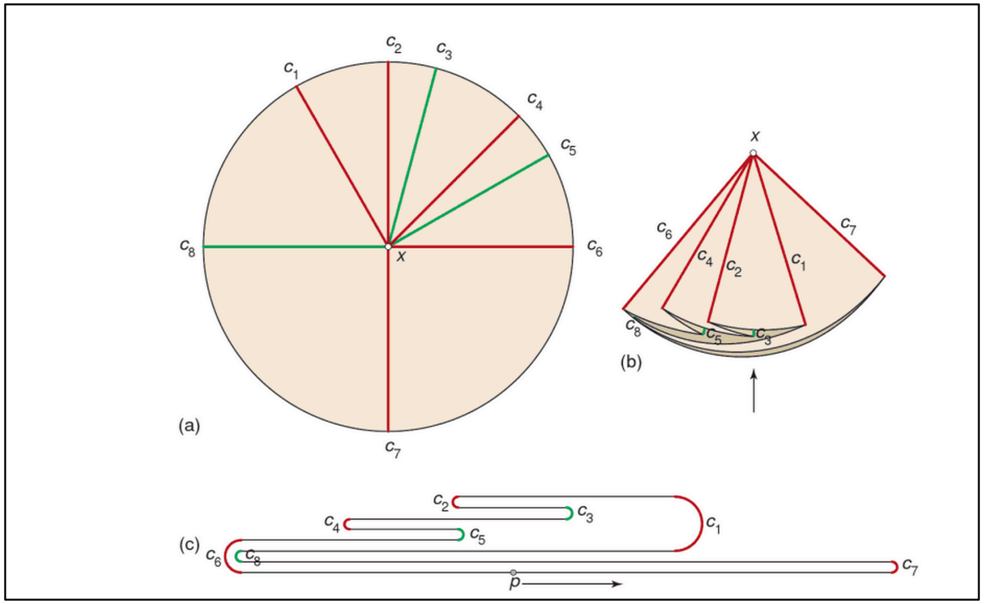
$$180\,(M - V) = 360$$

$$(M-V) = 2$$

$$M = V + 2$$

The Maekawa-Justin Theorem

If M mountain creases and V valley creases meet a vertex of a flat folding then M=V+2 or V=M+2



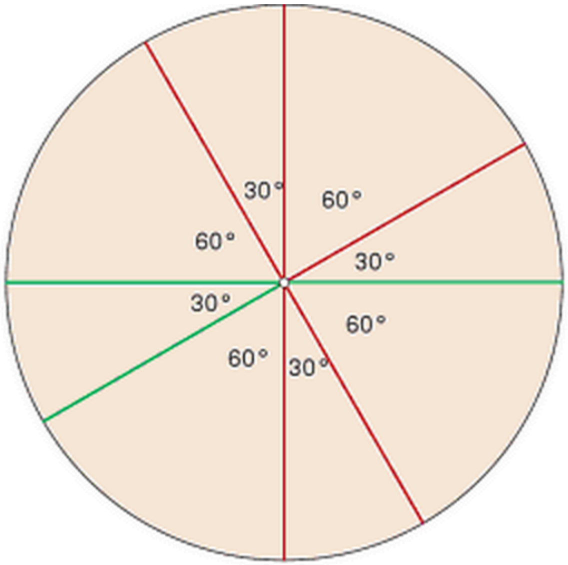
How do we know V=M+2 is possible?

What happens if you flip the paper?

The Maekawa-Justin Theorem

- Suppose I wrote a program to determine if the M-J Theorem holds for a sequence of **n** folds.
  About how many instructions would the program need to execute?

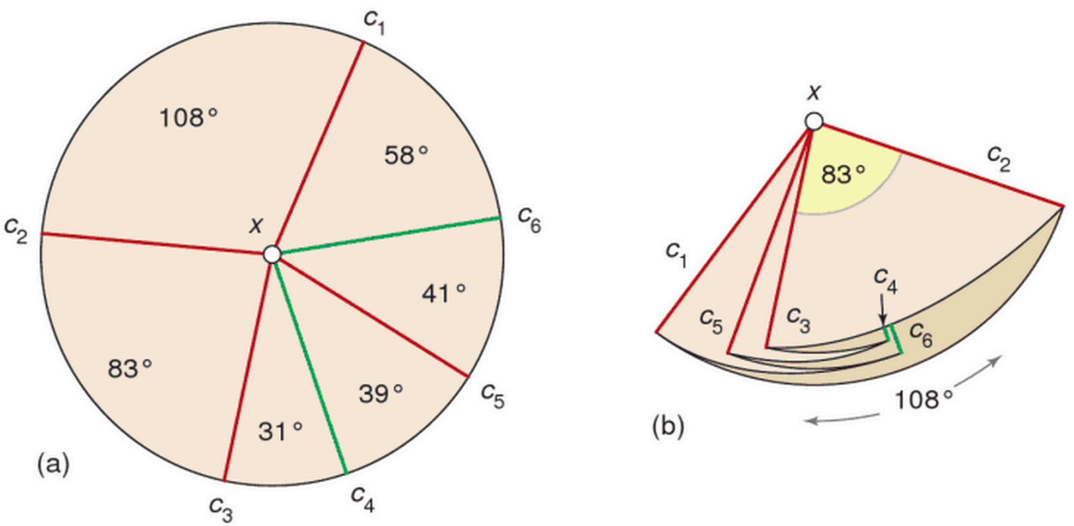
A Necessary Condition is not Enough



Is this crease pattern flat-foldable?

Does M-V = 2?

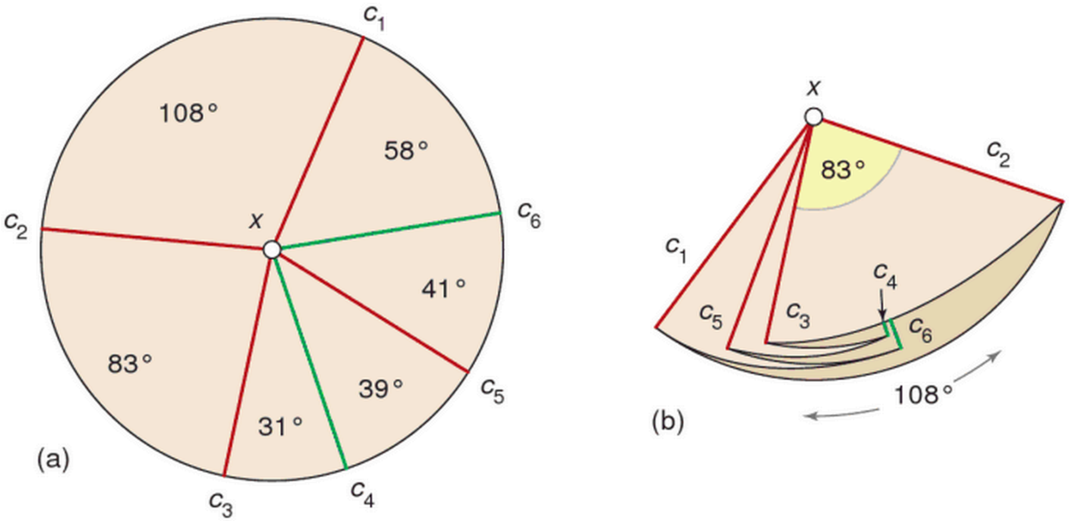
The Kawasaki-Justin Theorem



θ_1 __ θ_2 ___ θ_3 ___ θ_4 ___…+ θ_{n-1} ___ $\theta_n = 0°$

Can you fill in the blanks with + and/or – in a way that makes the statement true?

The Kawasaki-Justin Theorem

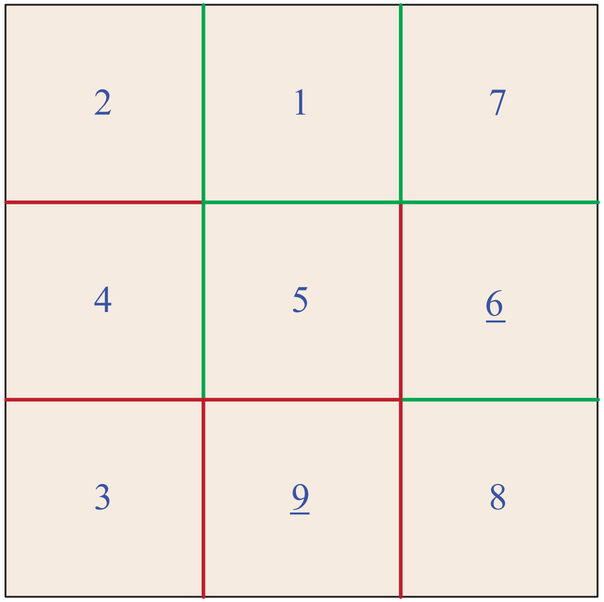


Let θ_i be the angles formed by creases incident to a vertex.

A set of an even number of creases that meet at a vertex folds flat if and only if $\theta_1 - \theta_2 + \theta_3 - \theta_4 + \ldots + \theta_{n-1} - \theta_n = 0°$

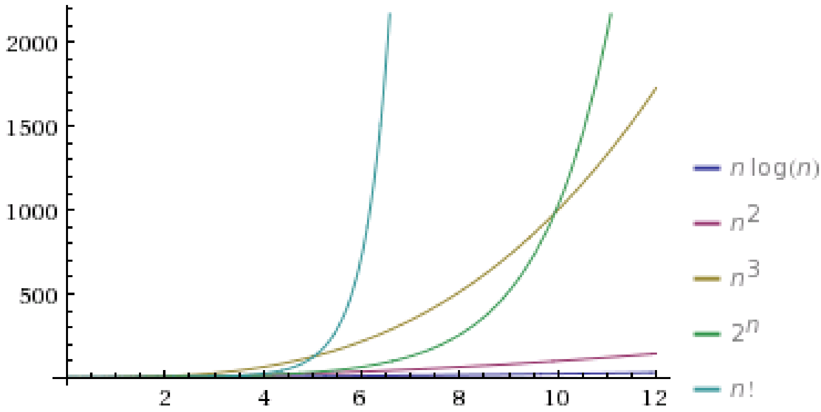
Do we know enough?

- Does what we have learned let me decide if this crease pattern is flat-foldable?



What does “hard to compute” mean?

- Informally, it means that the time required to compute the answer grows really rapidly as a function of the input
  - For example, if we have a problem that involves n creases and it takes 2^n instructions to find an answer
  - If the number of instructions is given by a polynomial function, like n^2, then the computation is generally not considered to be hard



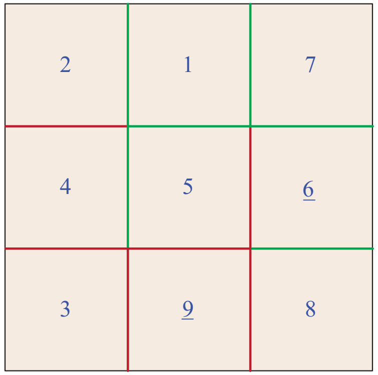
Graph taken from https://stackoverflow.com/questions/16388759/slowest-computational-complexity-big-o

Origami is hard

- There is no known algorithm for deciding foldability in polynomial time
- The problem is NP-Hard which means if there were a fast solution, we could solve a lot of other problems quickly as well…
- Finding a fast solution to an NP-Hard problem would one of the greatest mathematical discoveries. Ever.

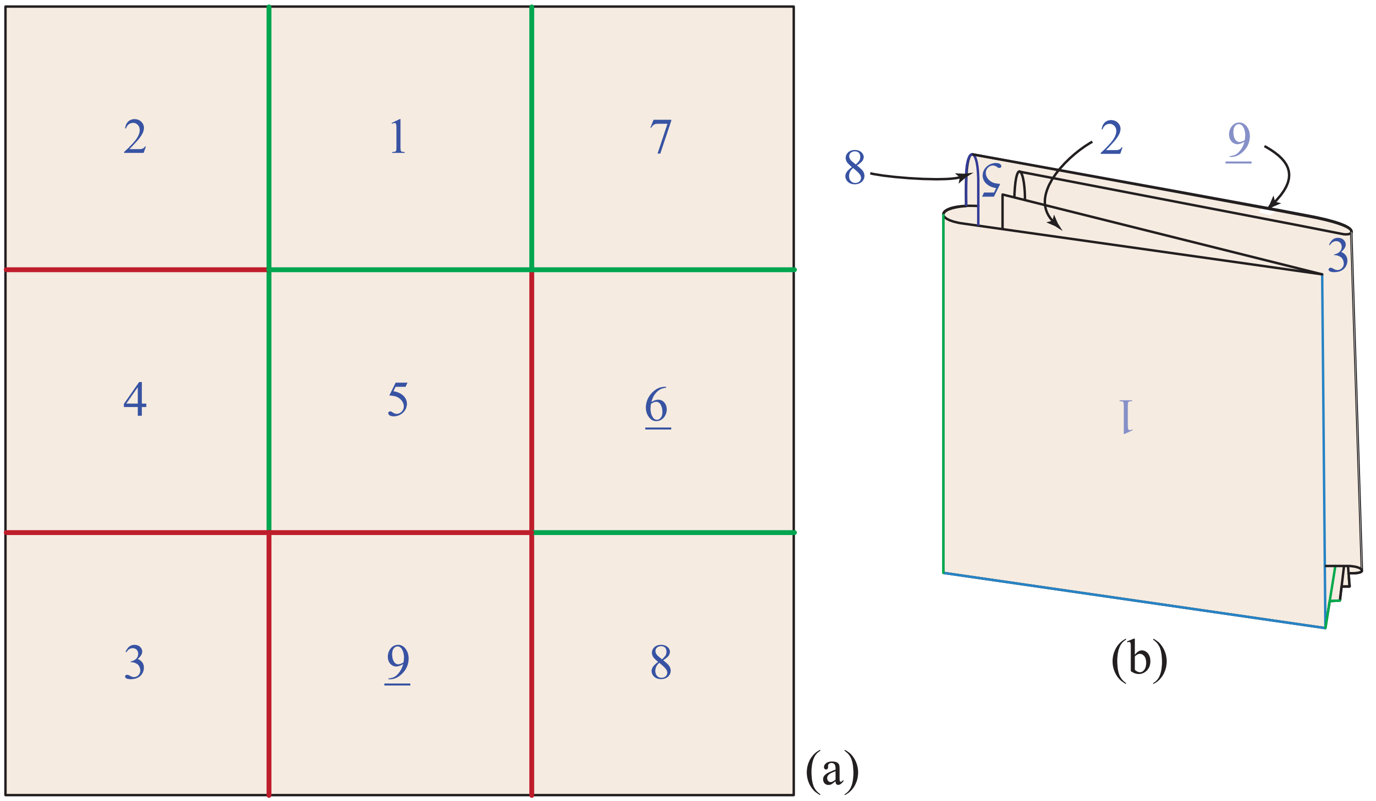
Open Problem

- An origami ***map*** is a regular grid of squares
  - each crease marked as a mountain or valley fold.



- Is there an efficient algorithm for determining if a given rectangular map can fold flat?
- Answer is unknown even for the **2 x n** case

And the solution…



Unattributed images and figures are taken from the excellent book:

How To Fold It: The Mathematics of Linkages, Origami, and Polyhedra.
By Joseph O'Rourke.
Cambridge University Press, 177 pages. ISBN 978-0521145473. 2011.